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On the first cohomology group for simply connected Lie groups

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Abstract. Let G be a simply connected Lie group, E be its Lie algebra, and ϕ be a continuous representation of G in a vector space V. First cohomology groups for both the group G and its Lie algebra E are shown to be isomorphic. When the group G is also semi-simple, the first cohomology group is shown to be trivial for a particular class of representations.

1. Introduction

In the theory of Lie groups, many global properties are more or less reflected in the infinitesimal ones. This reflection makes the Lie algebra an excellent tool for studying properties at large. An exception to this is that the connection between group cohomology (Eilenberg and MacLane 1947) and the Lie algebra cohomology (Hochschild and Serre 1953) with values in the representation space is not fully discussed. However, in the absence of a general method to compute this general cohomology, it is of great importance to present a study of low-order cohomology groups for continuous groups and their Lie algebras. Of particular interest are the first cohomology groups for certain Lie groups and their Lie algebras due to their physical relevance (Komy 1980, 1981). First cohomology groups of the Poincaré group relative to its various actions on sets of solutions of free relativistic wave equations are of great importance. With each cocycle found a fully relativistic quasi-free quantum field theory with finite energy can be constructed which will describe many attributes of the particles. The Poincaré group is a semi-direct product group and one must know first the corresponding cohomology groups for the semi-simple part. Also first-order cocycles were found to be of fundamental importance for the construction of continuous tensor products of group representations (Parthasarthy and Schmidt 1972). In § 2, the computation of the first cohomology groups for simply connected Lie groups is reduced to an algebraic one. Differential geometric notation will be used. If the group is also semi-simple, then its cohomology group for a certain class of representations is shown to be trivial.

2. First cohomology groups and their isomorphism

Let G be a simply connected Lie group, E be its Lie algebra, and ϕ is a continuous (unitary) representation in a complex Hilbert H. Let $\theta(=\mathrm{d}\phi)$ be the induced representation of E in H. In general the operators $\theta(X)$, $X \in E$ are not all bounded operators in H. Hence we have to consider only a proper common invariant dense domain for the set of unbounded operators. A possible choice for this domain is the Garding space of regular vectors of the representation ϕ. Sometimes it is more convenient to take as the invariant domain the space of analytic vectors for ϕ of G. This subspace (of H) is most convenient in applications and it is denoted by V (Barut and Raczka 1980). It can be shown that the first cohomology groups of the group G with values in H and respectively in V are isomorphic (Erven and Falkowski 1981).

A 1-cocycle for G is a smooth map f: $G \to V$, given by

$$f(xy) = \phi(x)f(y) + f(x), \qquad x, y \in G. \tag{1}$$

Each vector $a \in V$ determines a 1-cocycle given by,

$$f_a(x) = \phi(x)a - a, \qquad x \in G. \tag{2}$$

Such a 1-cocycle is called a 1-coboundary. Denote the set of 1-cocycles (resp. 1-coboundaries) by $Z^1(G, V)$ (resp. $B^1(G, V)$). Then the first cohomology group, denoted by $H^1(G, V)$, is defined as the quotient group

$$H^1(G, V) = Z^1(G, V)/B^1(G, V).$$

Similarly, a 1-cocycle for E is a linear map $F: E \to V$, given by the equation

$$F([h, k]) = \theta(h)F(k) - \theta(k)F(h), \qquad h, k \in E. \tag{3}$$

Also, each vector $a \in V$ determines a 1-cocycle F_a given by the equation

$$F_a(h) = \theta(h)a, \qquad h \in E. \tag{4}$$

Such a 1-cocycle is called a 1-coboundary. The first cohomology group, denoted by $H^1(E, V)$, is the quotient group

$$H^1(E, V) = Z^1(E, V)/B^1(E, V)$$

where $Z^1(E, V)$ (resp. $B^1(E, V)$) is the set of 1-cocycles (resp. 1-coboundaries).

The main result of this work is summarised by the following theorem.

Theorem 1. Let G be a simply connected Lie group, E be its Lie algebra, and ϕ is a continuous representation of G in a vector space V. Then we have the isomorphism,

$$H^1(G, V) \simeq H^1(E, V).$$

To prove this theorem we need the following lemma and propositions.

Lemma. Suppose that G is a connected group. Let f be a 1-cocycle such that $f'(e) = 0$. Then $f = 0$.

Proof. Since f is a 1-cocycle it follows from (1) that

$$f(e) = 0. \tag{5}$$

We now fix $b \in G$. Again, since f is a 1-cocycle we have

$$f(by) = \phi(b)f(y) + f(b), \qquad y \in G$$

which we write in the form

$$f(\lambda_b y) = \phi(b)f(y) + f(b), \qquad y \in G \tag{6}$$

where λ_b: G → G is a diffeomorphism. We then denote its derivative by $\mathrm{d}\lambda_b$ which is the bundle map (Greub *et al* 1973)

$$\mathrm{d}\lambda_b\colon T_G \to T_G.$$

We then differentiate (6) at the point $y = e$ to obtain

$$f'(b; \mathrm{d}\lambda_b(e, k)) = \phi(b) f'(e; k); \qquad k \in E.$$

Given that $f'(e) = 0$, it follows that $f(b) = \text{constant}$. Since G is connected, then $f(b) = f(e) = 0$, where relation (5) has been used. Since b is arbitrary, the lemma follows.

Proposition 1. Let $f \in Z^1(\mathrm{G}, V)$ and define the linear map $F\colon E \to V$ by setting

$$F(h) = f'(e; h), \qquad h \in E. \tag{7}$$

Then $F \in Z^1(E, V)$.

Proof. We fix $b \in \mathrm{G}$. Since f is a 1-cocycle, we have

$$f(xb) = \phi(x) f(b) + f(x), \qquad x \in \mathrm{G}$$

which we write in the form

$$f(\rho_b x) = \phi(x) f(b) + f(x), \qquad k \in \mathrm{G} \tag{8}$$

where ρ_b: G → G, is a diffeomorphism. We then differentiate (8) at the point $x = e$ to obtain

$$f'[b; \mathrm{d}\rho_b(e, h)] = \phi'(e; h) f(b) + f'(e; h), \qquad h \in E$$

which can be conveniently written in the form

$$Y_h(f) = \theta(h) f(b) + F(h), \qquad h \in E \tag{9}$$

where Y_h is the right invariant vector field generated by the vector h (Auslander 1963), and θ is as defined above. Next, let $k \in E$ be another vector and y_k is the corresponding right invariant vector field generated by k. Applying Y_k to equation (9), we easily deduce the following equation,

$$[Y_h, Y_k](f) = \theta(h) Y_k(f) - \theta(k) Y_h(f). \tag{10}$$

Since Y_h and Y_k are right invariant vector fields, we then have

$$[Y_h, Y_k](f) = Y_{[h,k]}(f).$$

Using this identity, equation (10) takes the form

$$Y_{[h,k]}(f) = \theta(h) Y_k(f) - \theta(k) Y_h(f)$$

which we write in the form

$$f'[b; Y_{[h,k]}(b)] = \theta(h) f'(b; k) - \theta(k) f'(b; h).$$

Setting $b = e$, and using equation (7), the last equation yields

$$F([h, k]) = \theta(h) F(k) - \theta(k) F(h), \qquad h, k \in E$$

i.e. $F \in Z^1(E, V)$, and the proposition follows.

Proposition 2. Suppose that G is a simply connected group. Let $F \in Z^1(E, V)$, then there exists an $f \in Z^1(G, V)$ such that $f'(e) = F$.

Proof. Consider the E-valued 1-form ω on G defined by

$$\omega[(X_h(x)] = \phi[x; F(h)], \qquad x \in G, h \in E \tag{11}$$

where X_h is the left parallel vector field determined by the vector h. It is convenient to write (11) in the form,

$$\omega(X_h) = \phi[(F(h)] = \phi(F_h). \tag{12}$$

It will be shown that ω is closed. For, let X_k be another left parallel vector field determined by the vector $k \in E$. If d is the exterior derivative (van Westenhoz 1978)

$$\mathrm{d}w(X_h, X_k) = X_h w(X_k) - X_k w(X_h) - w[[X_h, X_k]],$$

and on using definition (12), we obtain

$$\begin{aligned} \mathrm{d}\omega(X_h, X_k) &= (X_h\phi)F_k - (X_k\phi)F_h - \phi(F_{[h,k]}) \\ &= \phi\{\theta(h)F(k) - \theta(k)F(h) - F([h, k])\}. \end{aligned}$$

Since $F \in Z^1(E, V)$, then it follows that

$$\mathrm{d}\omega = 0.$$

Given that G is a simply connected group, the Poincaré inverse lemma ensures the existence of a smooth function (Helgason 1962), f: $G \to V$, such that $\omega = \mathrm{d}f$. We may choose the function f such that

$$f(e) = 0. \tag{13}$$

In particular,

$$f'(e; h) = \omega[(e; X_h(e)] = \phi(e)F(h) = F(h).$$

It remains to be shown that f is a 1-cocycle for G. We fix $b \in G$, and set

$$g(y) = f(by) - f(b) - \phi(b)f(y), \qquad y \in G. \tag{14}$$

We then put $y = e$, and use (13) to obtain $g(e) = 0$. Differentiating equation (14), we obtain

$$\begin{aligned} g'(y; k) &= f'(\lambda_b y, \mathrm{d}\lambda_b(y; k)) - \phi(b)f'(g'; f') \\ &= \omega[\lambda_b y; \mathrm{d}\lambda_b(y; k)] - \phi(b)w(y; k) \\ &= \phi(by)F(k) - \phi(b)\phi(y)F(k) = 0 \end{aligned}$$

where λ_b, $\mathrm{d}\lambda_b$ are the maps defined earlier, and $b \in E$.

The above equality yields $g(y) = \text{constant}$. Since G is a (simply) connected group then,

$$g(y) = g(e) = 0. \tag{15}$$

Equation (14) then yields,

$$f(by) = f(b) + \phi(b)f(y)$$

and the proposition follows.

Proof of theorem 1. Proposition 1 defines the linear map $\sigma: Z^1(G, V) \to Z^1(E, V)$ given by,

$$\alpha(f) = f'(e), \qquad f \in Z^1(G, V). \tag{16}$$

If $f \in B'(G, V)$, then it has the functional form, $f(x) = \phi(x)a - a$, for some $a \in V$. We differentiate this equation at the point $x = e$, and obtain

$$f'(e; h) = \theta(h)a, \qquad h \in E$$

and so $f \in B^1(G, V)$. Thus α restricts to the map

$$\alpha: B^1(G, V) \to B^1(E, V).$$

Therefore we have the induced map,

$$\alpha_{\#}: H^1(G, V) \to H^1(E, V).$$

In order to prove the theorem, we have to show that the induced map $\alpha_{\#}$ is an isomorphism.

Let $[F] \in H^1(E, V)$ be the class of cohomologous 1-cocycles for E, and let $F \in Z^1(E, V)$ represent this class. By proposition 2, there exists an $f \in Z^1(G, V)$, such that

$$\alpha(f) = F.$$

Let $[f] \in H^1(G, V)$ be the class of cohomologous 1-cocycles for G, then we have

$$\alpha_{\#}[f] = [\alpha(f)] = [F]$$

and so $\alpha_{\#}$ is onto. Next consider the element $f \in Z^1(G, V)$, such that $\alpha(f) \in \phi^1(E, V)$, then we have

$$f'(e; h) = \theta(h)a, \qquad \text{for some } a \in V, \text{ and } H \in E. \tag{17}$$

Now, define the function $g: G \to V$, by the relation:

$$g(x) = f(x) - \phi(x)a + a, \qquad X \in G, \text{ and } a \in V. \tag{18}$$

Hence $g \in Z^1(G, V)$. Differentiate (18) at point $x = e$ to obtain,

$$g'(e; h) = f'(e; h) - \phi'(e; h)a = \theta(h)a - \theta(h)a = 0, \qquad h \in E. \tag{19}$$

Since G is connected, using the lemma, equation (19) yields:

$$g(x) = 0, \qquad x \in G.$$

Then

$$f(x) = \phi(x)a - a$$

i.e. $f \in B^1(G, V)$, and so $\alpha_{\#}$ is $1-1$, and the theorem follows.

3. Computation of $H^1(E, V)$

To compute $H^1(G, V)$ for simply connected Lie groups, and in view of theorem 1, we have to compute $H^1(E, V)$ for the Lie algebra E. Since this cohomology is representation dependent, different classes of representations must be considered separately. An important case is when the Lie algebra E is semi-simple. For a certain class of representations we have the following theorem.

Theorem 2. Let E be a semi-simple Lie algebra, and θ is a representation for E in a vector space V such that second order Casimir operator is invertible, then $H^1(E, V)=0$.

Proof. Let $\{e_\alpha\}_1^n$ be a basis for E, then

$$[e_\alpha, e_\beta]=C^\gamma_{\alpha\beta}e_\gamma, \qquad \alpha, \beta, \gamma=1,2,\ldots \tag{20}$$

wt.ere $C^\gamma_{\alpha\beta}$ are the structure constants. Since θ is a representation for E, we have

$$[\theta(e_\alpha), \theta(e_\beta)]=C^\delta_{\alpha\beta}\theta(e_\gamma), \qquad \alpha, \beta, \gamma=1,2,\ldots, n. \tag{21}$$

The second-order Casimir operator (exists only for semi-simple Lie algebra) is defined as a function Ω on V by the relation

$$\Omega=g^{\alpha\beta}\theta(e_\alpha)\theta(e_\beta), \qquad \alpha, \beta=1,2,\ldots, n$$

where $g_{\alpha\beta}$ is the metric tensor (or, the Killing form). Using relation (21), it is easy to show that,

$$\Omega\theta(e_\alpha)=\theta(e_\alpha)\Omega, \qquad \alpha=1,2,\ldots, n. \tag{22}$$

Since F is a 1-cocycle for E, and if we set $h=e_\alpha$ and $k=e_\beta$, then equation (3) takes the form

$$C^\gamma_{\alpha\beta}F(e_\gamma)=\theta(e_\alpha)F(e_\beta)-\theta(e_\beta)F(e_\alpha), \qquad \alpha, \beta=1,2,\ldots n. \tag{23}$$

Next, define the element $b\in V$ by the relation

$$b=g^{\alpha\beta}\theta(e_\alpha)F(e_\beta), \qquad \alpha, \beta=1,2,\ldots, n. \tag{24}$$

From (23) and (24) we deduce that:

$$\theta(e_\alpha)b=[g^{\mu\lambda}c^\gamma_{\alpha\mu}+g^{\mu\delta}c^\lambda_{\alpha\mu}]\theta(e_\gamma)F(e_\lambda)+\Omega F(e_\alpha). \tag{25}$$

Since E is semi-simple, then the invariance of the Killing form gives the relation

$$g^{\mu\lambda}c^\gamma_{\alpha\mu}+g^{\mu\nu}c^\lambda_{\alpha\mu}=0$$

hence equation (25) reduces to

$$\theta(e_\alpha)b=\Omega F(e_\alpha), \qquad \alpha=1,2,\ldots, n.$$

Given that Ω is invertible, i.e. Ω^{-1} exist, then the above relation yields

$$\theta(e_\alpha)\Omega^{-1}b=F(e_\alpha), \qquad \alpha=1,2,\ldots, n. \tag{26}$$

The element $\Omega^{-1}b$ is an element in V, say a, hence (26) reduces to

$$F(e_\alpha)=\theta(e_\alpha)a, \qquad \alpha=1,2,\ldots, n.$$

However the linearity of F implies that,

$$F(h)=\theta(h)a; \qquad \forall h\in E.$$

This is the functional relation (4), and the theorem follows.

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